HOME WORK 2, PROBABILITY II. MARKOV CHAINS.

1. Show that a simple random walk on a graph is irreducible if and only if the graph is connected. (recall that a graph is called connected if for every pair of vertices x and y there exists a sequence of vertices $v_1, ..., v_k$ such that $(x, v_1), (v_1, v_2), ..., (v_{k-1}, v_k), (v_k, y) \in E$, where E is the set of edges.)

2. Let P be an irreducible transition matrix of period p. Show that Ω can be expressed as a disjoint union of p sets $\Omega_1, ..., \Omega_p$ so that $P(x, y) > 0$ only if $x \in \Omega_j$ and $y \in \Omega_{j+1 \mod p}$, for some integer j between 1 and p.

3. Let $p \in (0, 1)$. Suppose a gambler has k dollars, where k is a positive integer. They place a bet of \$1 with probability p every minute, and don't do anything with probability $1 - p$. The gambler wins with probability $\frac{1}{2}$, and loses with the same probability. The gambler stops playing if either they run out of money, or they get n dollars, where $n \geq k$. Find the expected number of minutes the game will last.

4. Find the stationary distribution for the Ehrenfest chain.

5. Given a transition matrix of a Markov chain P on Ω, and a pair of distributions μ_1 and μ_2 on Ω, show that the total variation distance between $\mu_1 P$ and $\mu_2 P$ is less than or equal to the total variation distance between μ_1 and μ_2.

6. There is a line at a grocery shop. Unless the line is empty, every second either a new customer arrives with probability α, or a customer is served with probability $1 - \alpha$. If the line is empty, then a new customer still arrives with probability α, and with probability $1 - \alpha$ nothing happens. Let X_t be the number of customers in the line at time t.

a) Depending on α, find whether X_t is positive recurrent, null recurrent or transient.

b) In the positive recurrent case, determine the stationary distribution.

c) In the positive recurrent case, find the expected time in which a customer is served after his or her arrival, provided that the arrival occurs when the distribution is stationary.